SOLVING THE EQUATIONS OF A LAMINAR BOUNDARY LAYER UNDER ARBITRARY INITIAL CONDITIONS

(O RESHENII URAVNENII LAMINARNOGO POGRANICHNOGO SLOIA PRI PROIZVOL'NYKH NACHAL'NYKH USLOVIIAKH)

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V.Ia.NEILAND
(Moscow)

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The study of the asymptotic behavior of the solutions of Navier-Stokes equations for large values of the Reynolds number necessitates the integration of the Prandtl boundary layer equations for nonuniform velocity and enthalpy profiles in the initial section or on bodies extending to infinity upward and downward along the flow.

The present paper offers transformations which make it possible to reduce these problems to a form permitting the use of known self-similar solutions and well-developed numerical [1 and 2] or analytic [3 and 4] methods for the integration of boundary layer equations on finite or semi-infinite bodies in uniform free flows.

The first Section is an investigation of a laminar boundary layer in a compressible gas with nonuniform velocity and enthalpy profiles in the initial cross section. The second section concerns the problem of the boundary layer on a body extending to infinity upward and downward along the flow.

1. The equations of a two-dimensional laminar layer in a compressible gas can be conveniently considered in the form proposed in [5] (Dorodnitsyn -Lees variables)

$$(Nf'')' + ff'' + 2\frac{\xi}{u_0}u_0^{\cdot}\left(\frac{\rho_0}{\rho} - f'^2\right) = 2\xi\,(f'f'^{\cdot} - f^{\cdot}f''), \quad N = \frac{\rho\mu}{(\rho\mu)_w} \qquad (1.1)$$

$$\left(\frac{N}{\sigma}g'\right)' + fg' + \frac{u_0^2}{H_0}\left[N\left(1 - \frac{1}{\sigma}\right)f'f''\right]' = 2\xi\,(f'g^{\cdot} - f^{\cdot}g'), \quad u = u_0 f'(\xi, \eta) \qquad (1.2)$$

$$\xi = \xi^\circ + \int_0^x (\rho\mu)_w u_0 r^{2j}dx, \qquad \eta = \frac{u_0 r_w^{\,j}}{\sqrt{2\xi}}\int_0^y \rho\, dy, \qquad H = H_0 g(\xi, \eta) \qquad (1.3)$$

where x is the coordinate measured along the surface of the body from the initial point of integration; the subscripts 0 and w denote values of quantities on the outer boundary of the boundary layer and at the surface of the body: σ is the Prandtl number; $j = 0$ or 1 for plane and axisymmetrical flows, respectively; H is the stagnation enthalpy; u is the velocity component along the x-axis; the prime denotes differentiation with respect to η , the dot differentiation with respect to ξ .

The distinction from the usual form of the variables consists in the fact that the quantity $\xi^\circ \neq 0$ in Formula (1.3). This is necessary, since for $x = 0$ we are given the initial conditions

$$f(\xi^\circ, \eta) = \chi(\eta), \qquad g(\xi^\circ, \eta) = \psi(\eta) \tag{1.4}$$

If, as usual, $\xi^\circ = 0$, then by Formula (1.3), the quantity η goes to infinity at $x^\circ = 0$ for small but finite y .

As regards the function $\chi(\eta)$ we assume that the leading term in its expansion as $\eta \to 0$ can be represented in the form

$$\chi(\eta) \sim \eta^n \tag{1.5}$$

where n is some constant. It is clear that the point ($\xi = \xi^\circ$, $\eta = 0$) is generally singular for the problem which we have formulated.

In integrating the boundary layer equations near this singular point we cannot make use of system (1.1),(1.2), since the application of any numerical method here would require infinite condensation of the computational grid as the singularity was approached.

In order to eliminate this difficulty, we propose the use of the following variables in the neighborhood of the singular point: (1.6)

$$\xi_1 = \int_0^x (\rho\mu)_w u_0 r^{2j} dx, \qquad \eta_1 = \eta \left(\frac{\xi}{\xi_1}\right)^{\frac{1}{n+2}}, \qquad \varphi(\xi_1, \eta_1) = \left(\frac{\xi}{\xi_1}\right)^{\frac{n+1}{n+2}} f(\xi, \eta)$$

$$h(\xi_1, \eta_1) = g(\xi, \eta)$$

Upon conversion to these variables, Equations (1.1) and (1.2) become

$$(N\varphi'')' + \frac{1}{\xi}\left[\xi_1 + \frac{2(n+1)}{n+2}\xi^\circ\right]\varphi\varphi'' - \frac{2n}{n+2}\frac{\xi^\circ}{\xi}\varphi'^2 + $$
$$+ 2\frac{d\ln u_0}{d\ln \xi_1}\left[\left(\frac{\xi_1}{\xi}\right)^{-\frac{2n}{n+2}}\frac{\rho_0}{\rho} - \varphi'^2\right] - 2\xi_1(\varphi'\dot\varphi' - \dot\varphi\varphi'') = 0 \tag{1.7}$$

$$\left(\frac{Nh'}{\sigma}\right)' + \frac{1}{\xi}\left[\xi_1 + \frac{2(n+1)}{n+2}\xi^\circ\right]\varphi h' + \frac{u_0^2}{H_0}\left(\frac{\xi_1}{\xi}\right)^{\frac{2n}{n+2}}\left[N\left(1 - \frac{1}{\sigma}\right)\varphi'\varphi''\right]' - $$
$$- 2\xi_1(\varphi'\dot h - \dot\varphi h') = 0 \tag{1.8}$$

The prime here denotes differentiation with respect to η_1 and the dot with respect to ξ_1 .

It is easy to see that for $\xi_1 = 0$ one can arrive at a solution which depends solely on η_1 . The boundary conditions for $\eta_1 = 0$ are the usual ones. Formula (1.6) indicates that the corresponding value of η_1 tends to infinity with $\xi_1 = 0$ for arbitrarily small but finite values of η . This means that in determining the boundary conditions on the outer boundary $\eta_1 \to \infty$ for system (1.7),(1.8) one ought to take the corresponding values of the variables in the specified initial profile for $\eta = 0$ (i.e. at the wall).

$$\xi_1 = 0, \quad h(0, \infty) = g(\xi^\circ, 0), \quad \lim_{\eta_1 \to \infty} \varphi'(0, \eta_1) \sim \lim_{\eta \to 0} \chi(\eta)$$

Specifically, if the velocity for $\eta = 0$ in the initial profile is not zero, then $n = 0$, and the velocity at the outer boundary of the inner layer is equal to the velocity at the wall in the external profile

$$\varphi'(0, \infty) = f'(\xi^\circ, 0)$$

We note also that Equations (1.7) and (1.8) in this case coincide in form with Equations (1.1) and (1.2). A local solution is clearly the solution for a flat plate or for the corresponding flow with a variable velocity at the outer boundary.

Let us also consider the case $n = 1$, $u_0 =$ const. For $\xi_1 = 0$, the dissipative term in the energy equation vanishes for all $n \neq 0$; this is quite understandable from the physical standpoint. The momentum equation and boundary conditions for $n = 1$ and $N \equiv 1$ become

$$\varphi''' + \frac{4}{3}\varphi\varphi'' - \frac{2}{3}\varphi'^2 = 0, \qquad \lim \varphi'' = f''(\xi, 0) \quad \text{for } \eta_1 \to \infty$$

The solution of this problem is the local Couette solution

$$\varphi''(0, \eta_1) = f''(\xi^\circ, 0) = \text{const}$$

Thus, initial profiles (1.4) are specified on the first characteristic in the outer part. Near the singular point the problem does not require initial conditions and reduces, as in the conventional case, to the integration of a system of ordinary differential equations with boundary conditions specified on different boundaries of the region.

The boundary conditions on the body surface for a nonpermeable surface at a specified temperature are of the usual form.

$$\varphi(0, 0) \neq \varphi'(0, 0) = 0, \qquad h(0, 0) = h_w$$

Making use of Formulas (1.6), one can readily write out the boundary conditions for other cases.

On the other characteristics $\xi =$ const or $\xi_1 =$ const integration can be carried out over any variables. It is convenient, however, to pick some value $\eta_1 = \eta_1^\circ$ and to integrate for $\eta_1 \leq \eta_1^\circ$ with the aid of variables (1.6), and above this, with the aid of the ordinary variables. Conversion from the lower system to the upper is effected by means of Formulas (1.6). Such a procedure automatically guarantees uniform accuracy of the solution without condensation of the computational grid (or without any increase in the accuracy of interpolation) near the surface of the body.

The boundary conditions on the outer boundary of the boundary layer are then required only for the outer system (1.1),(1.2) and have the usual form

$$f'(\xi, \infty) = g(\xi, \infty) = 1$$

The quantity η_1° is practically chosen to correspond to infinity for the inner sublayer with $\xi_1 = 0$. Its value is approximately equal to 5 to 8, and of course depends on the problem under consideration.

The longitudinal coordinates ξ and ξ_1 are related by Expression

$$\xi = \xi_1 + \xi^\circ$$

where ξ° is a specified constant. It is clear that as one moves away from the singular point the ratio (ξ/ξ_1) tends to unity. For this reason transformations (1.6) lose their special character. Furthermore, the new variables asymptotically approach the old ones: $\varphi \to f$, $\eta_1 \to \eta$, and all terms and coefficients which distinguish them from Equations (1.1) and (1.2) disappear from Equations (1.7) and (1.8).

2. Boundary layers are usually considered on finite or semi-infinite bodies. However, the solution of certain problems whose consideration exceeds the scope of the present paper requires computation of the laminar boundary layer on a body of infinite length extending upward and downward along the flow. It is evident that in most cases the thickness of such a layer at points with finite coordinates must become infinitely large. On the other hand, in the case of accelerating flows whose velocity at a point far removed upward along the flow toward infinity is equal to zero, it is indeed possible to find solutions which do have physical meaning.

Let us consider the equations of a compressible two-dimensional laminar boundary layer. We introduce new independent variables in the form

$$\xi = \int_{-\infty}^{x} (\rho\mu)_w F(u_0)\, dx, \qquad \eta = \Phi(u_0) \int_0^y \rho\, dy \tag{2.1}$$

The functions $F(u_0)$ and $\Phi(u_0)$ are as yet undetermined. It s necessary, however, that $F(u_0)$ be such that integral (2.1) converges as $x \to -\infty$.

Upon conversion to variables (2.1), the momentum equation becomes

$$(Nf'')' + \alpha f f'' + 2\beta\left(\frac{\rho_0}{\rho} - f'^2\right) = \frac{u_0 F(u_0)}{\Phi^2(u_0)} (f'\dot{f}' - \dot{f} f'') \tag{2.2}$$

$$\alpha = \frac{F(u_0)}{\Phi(u_0)} \frac{d}{d\xi}\left[\frac{u_0}{\Phi(u_0)}\right], \qquad 2\beta = \frac{F(u_0)}{\Phi^2(u_0)} \frac{du_0}{d\xi} \tag{2.3}$$

An infinitely distant point corresponds to the value $\xi = 0$. In order for it to be possible to begin integration from the point $\xi = 0$, it is necessary that the coefficient in the right-hand side of Equation (2.2) to vanish for $\xi = 0$. In order to reduce Equation (2.2) to the usual form, we require fulfillment of the following conditions:

$$u_0 F(u_0) = 2\xi\Phi^2(u_0), \qquad \alpha = \text{const}, \qquad \beta = \text{const} \qquad (x \to -\infty) \tag{2.4}$$

The solution of system (2.3),(2.4) is of the form

$$u_0 = c\xi^\beta \qquad \Phi = A u_0^{\frac{2\beta-\alpha}{2\beta}}, \qquad F = 2A^2 c^{-\frac{1}{\beta}} u_0^{1+\frac{1-\alpha}{\beta}} \tag{2.5}$$

The case $\beta = 0$ corresponding to $u_0 = \text{const}$ is of no interest in connection with the present problem.

The above solutions and Formula (2.1) indicate that in the neighborhood of the point $x \to -\infty$ the dependence $u_0(x)$ must be of the form

$$2\beta A^2 \int (\rho\mu)_w\, dx + \text{const} = \begin{cases} \ln u_0 & \text{for } \alpha = \beta \\ [\beta/(\alpha-\beta)]\, u_0^{(\alpha-\beta)/\beta} & \text{for } \alpha \neq \beta \end{cases} \tag{2.6}$$

Making use of the above Formulas, we can easily show that with $\alpha = \beta$ we can assume that $\alpha = 1$ and set

$$\xi = \frac{u_0(0)}{c} \exp\left[2A^2 \int_0^x (\rho\mu)_w \, dx\right] \tag{2.7}$$

It is evident that as $x \to -\infty$ we obtain $\xi = 0$. The velocity distribution for the corresponding self-similar solution is of the form

$$u_0 = u_0(0) \exp\left[2A^2 \int_0^x (\rho\mu)_w \, dx\right] \tag{2.8}$$

The second variant of solution (2.6) for $\alpha \neq \beta$ leads to the following formula for the velocity:

$$u_0 = \left[u_0(0)^{\frac{\alpha-\beta}{\beta}} - 2(\beta - \alpha) A^2 \int_0^x (\rho\mu)_w \, dx\right]^{-\frac{\beta}{\beta-\alpha}} \tag{2.9}$$

In order for $u_0(-\infty)$ to vanish it is sufficient that $\beta > 0$ and $\beta > \alpha$. In the neighborhood of an infinitely distant point the integrand in (2.1) can be estimated by means of Formula

$$(\rho\mu)_w F(u_0) \sim x^{-\frac{1}{\beta-\alpha} - 1} \tag{2.10}$$

Estimate (2.10) indicates that integral (2.1) converges as $x \to -\infty$.

It should be noted that in those cases where $u_0 \sim (-x)^n$ for $0 < n < 1$ it is necessary to set $\alpha < 0$. This follows from Formula (2.9) and from the limitation $\beta > 0$.

Heretofore we have confined our attention to the momentum equation. As for the energy equation, the coefficients of almost all terms are the same as those considered above. The only difference lies in the coefficient of the so-called dissipative term, which is equal to u_0^2/H_0. However, by virtue of the assumptions made, the velocity vanishes at an infinitely distant point. This causes the dissipative term to vanish.

We have therefore shown that the computation of the laminar boundary layer on an infinite body the flow past which accelerates with laws (2.8) and (2.9) describing the velocity distribution in its outer stream reduces to the computation of an ordinary laminar boundary layer on a finite or semi-infinite body using transformations of the type (2.1).

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